# A Short Note on Exact Equality for the Scholz-Brauer Conjecture

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## Definitions

An addition chain is a finite list of integers starting with 1 where each value (called an element) apart from the first is the sum of two prior elements:

$1 = a_0 < a_1 < \cdots < a_r = n, a_i = a_j + a_k, i > j \geq k \geq 0$. We say this addition chain has length $r$ for target $n$. The shortest addition chain for $n$ is said to have length $l(n)$.

We define the following useful functions:

$$v(n) = \begin{cases} 0 & n = 0 \\ v(\frac{n-1}{2}) + 1 & n \text{ odd} \\ v(\frac{n}{2}) & n \text{ even} \end{cases}, \ \lambda(n) = \lfloor \log_2(n) \rfloor, \ s(n) = l(n) - \lambda(n)$$

We can extend some of these functions to sequences of integers. This is known as the addition sequence problem. For a set $A$ we define $l(A)$ as the shortest chain that contains all elements of $A$. $\lambda(A) = \lambda(max(A))$ and $s(A) = l(A) - \lambda(A)$.

## Conjectures

This paper concerns the Scholz-Brauer conjecture $l(2^n - 1) \leq l(n) + n - 1$ [4] and more precisely a violation of equality in this conjecture $l(2^n - 1) < l(n) + n - 1$. The Knuth-Stolarsky conjecture is related and bounds $v(n)$ with the small step count of $n$ denoted $s(n)$ such that $v(n) \leq 2^{s(n)}$. For a fixed $\lambda$ obviously $2^\lambda - 1$ has the largest $v$ (binary digit sum) values so this conjecture implies a lower bound for $l(2^n - 1)$. Now we can restate this conjecture as $s(2^n - 1) \leq l(n)$ and in fact this is useful for what will follow.

## History of the Conjecture

Stolarsky [5] seems to have first noted equality for $1 \leq n \leq 8$. Knuth [3] noted equality for $n \leq 11$ and asked if equality always held in the last question of §4.6.3. Thurber [6] attributed the range $n \leq 14$ to Knuth and proved equality for $n \leq 18$ along with $n \in \{20, 24, 32\}$. Flammenkamp [2] extended this to

$n \leq 28$ although the assumed the conjecture $v(n) \leq 2^{s(n)}$. My own calculations [1] extended this to $n \leq 64$. Later code would extend this to $l(n) \leq 9$ so that includes $n \leq 126$.

## Using the Factorization of $2^n - 1$

Since $l(a \cdot b) \leq l(a) + l(b)$ factorization might be an approach to find counterexamples. If $n$ is composite $n = a \cdot b, a > 1$ then $2^n - 1 = (2^a - 1)(2^{a \cdot b - a} + ... + 2^a + 1)$. There doesn't seem much scope here since the first factor $(2^a - 1)$ is in the same form as the number we are trying to find a counterexample with. The second term has single bits spread out in a way that makes it hard to envision carries helping construct in shorter sequences. If $n$ is prime then the factors I looked at tended to have large hamming weights.

I decided then to look at non-unique factorizations. We use $2^n - 1 = \sum_{i=1}^{z} a_i \cdot x_i$ since this has been shown to be the structure of all shortest addition chains [7]. Here $A = a_1, ..., a_z$ and $X = x_1, ..., x_z$ are two addition sequences such that $l(2^n - 1) \leq l(A) + l(X) + z - 1$. The bound is sharp for at least one selection of $A, X$.

One possible strategy for selecting a sequence $A$ is to enumerate all shortest addition chains for some $2^m - 1, m < n$. We can select the elements of $A$ based on some fitness for building $2^n - 1$. The strategy used was to look at all elements of the form $2^{d_i}(2^{b_i} - 1)$. We can extract the sequence of $B = b_1, ..., b_z$ values from the addition chain for $2^m - 1$ and look for sequences $B$ where $l(B) > s(2^m - 1)$. It's not at all clear that such sequences should exist but a large amount of computation shows that they do. The smallest example is in the shortest addition chains for $2^{47} - 1$:

1, 2, 3, 6, 12, 24, 30, 60, 120, 240, 480, 483, 963, 1926, 3852, 7704, 8187, 8189, 16376, 32752, 65504, 131008, 262016, 524032, 1048064, 2096128, 4192256, 8384512, 16769024, 16777213, 16777215, 33554428, 67108856, 134217712, 268435424, 536870848, 1073741696, 2147483392, 4294966784, 8589933568, 17179867136, 34359734272, 68719468544, 137438937088, 274877874176, 549755748352, 1099511496704, 2199022993408, 4398045986816, 8796091973632, 17592183947264, 35184367894528, 70368735789056, 140737471578112, 140737488355327

Here by abusing the notation we can select $B \subseteq \{1, 2, 4, 11, 23, 24, 47\}$ and hence $A \subseteq \{2^1 - 1, 2^2 - 1, 2 \cdot (2^4 - 1), 8 \cdot (2^{11} - 1), 4 \cdot (2^{23} - 1), 2^{24} - 1, 2^{47} - 1\}$. Note here that $s(2^{47} - 1) = l(47) = 8$ but $l(\{1, 2, 4, 11, 23, 24, 47\}) = 9$. We should look for numbers $n$ where $l(n) > l(B) + l(Y) - z + 1$ where $Y = y_1, ..., y_z$ and $n = \sum_{i=1}^{z} b_i \cdot y_i$. While we can't use the $B$ values to create a shortest addition chain for $n$ we can use the $A$ values derived from them to try and form an addition chains for $2^n - 1$.

Since the crouton algorithm for addition chains works by enumerating sequences like $B$ and then using what it can infer about the values in $Y$ to prune. We can find example numbers very quickly even over very large ranges.

We find the addition chain:

1, 2, ?, ?, 11 ?, 23, 24, 47, 71, 142, 284, 568, 1136, 2272, 4544, 9088, 18176, 36352, 36363, 72715, 145430, 290860, 581720, 1163440, 2326880, 4653760, 4653783,

9307543
Here we leave the uncertainty about the start of the chain for $n$ as question marks. We then convert this to a chain for $2^{9307543} - 1$ by starting with the special chain for $2^{47} - 1$ we showed earlier.
The following table shows the breakdown:

| Elements | How First Element Formed | Elements In Row | Chain Length So Far |
|---|---|---|---|
| 1 | | 1 | 0 |
| 2 | $1+1$ | 1 | 1 |
| 3 | $2+1$ | 1 | 2 |
| $6, ..., 24$ | $3+3$ | 3 | 5 |
| $2\cdot(2^4-1), ..., 2^5\cdot(2^4-1)$ | $24+6$ | 5 | 10 |
| 483 | $2^5\cdot\ (2^4-1)+3$ | 1 | 11 |
| $963, ..., 2^3\cdot 963$ | $483+2^5\cdot(2^4-1)$ | 4 | 15 |
| 8187 | $2^3\cdot 963+483$ | 1 | 16 |
| 8189 | $8187+2$ | 1 | 17 |
| $2^3\cdot(2^{11}-1), ..., 2^{13}\cdot(2^{11}-1)$ | $8189+8187$ | 11 | 28 |
| 16777213 | $2^{13}\cdot(2^{11}-1)+8189$ | 1 | 29 |
| $2^{24}-1$ | $16777213+2$ | 1 | 30 |
| $2^2\cdot(2^{23}-1), ..., 2^{24}\cdot(2^{23}-1)$ | $2^{24}-1+16777213$ | 23 | 53 |
| $2^{47}-1, ..., 2^{24}\cdot(2^{47}-1)$ | $2^{24}\cdot(2^{23}-1)+2^{24}-1$ | 25 | 78 |
| $2^{71}-1, ..., 2^{71}\cdot(2^{71}-1)$ | $2^{24}\cdot(2^{47}-1)+2^{24}-1$ | 72 | 150 |
| $2^{142}-1, ..., 2^{142}\cdot(2^{142}-1)$ | $2^{71}\cdot(2^{71}-1)+2^{71}-1$ | 143 | 293 |
| $2^{284}-1, ..., 2^{284}\cdot(2^{284}-1)$ | $2^{142}\cdot(2^{142}-1)+2^{142}-1$ | 285 | 578 |
| $2^{568}-1, ..., 2^{568}\cdot(2^{568}-1)$ | $2^{284}\cdot(2^{284}-1)+2^{284}-1$ | 569 | 1147 |
| $2^{1136}-1, ..., 2^{1136}\cdot(2^{1136}-1)$ | $2^{568}\cdot(2^{568}-1)+2^{568}-1$ | 1137 | 2284 |
| $2^{2272}-1, ..., 2^{2272}\cdot(2^{2272}-1)$ | $2^{1136}\cdot(2^{1136}-1)+2^{1136}-1$ | 2273 | 4557 |
| $2^{4544}-1, ..., 2^{4544}\cdot(2^{4544}-1)$ | $2^{2272}\cdot(2^{2272}-1)+2^{2272}-1$ | 4545 | 9102 |
| $2^{9088}-1, ..., 2^{9088}\cdot(2^{9088}-1)$ | $2^{4544}\cdot(2^{4544}-1)+2^{4544}-1$ | 9089 | 18191 |
| $2^{18176}-1, ..., 2^{18176}\cdot(2^{18176}-1)$ | $2^{9088}(2^{9088}-1)+\ 2^{9088}-1$ | 18177 | 36368 |
| $2^{36352}-1, ..., 2^{14}\cdot(2^{36352}-1)$ | $2^{18176}\cdot(2^{18176}-1)+2^{18176}-1$ | 15 | 36383 |
| $2^3(2^{36363}-1), ..., 2^{36352}(2^{36363}-1)$ | $2^{14}\cdot(2^{36352}-1)+\ 2^3\cdot(2^{11}-1)$ | 36350 | 72733 |
| $2^{72715}-1, ..., 2^{72715}\cdot(2^{72715}-1)$ | $2^{36352}\cdot(2^{36363}-1)+2^{36352}-1$ | 72716 | 145449 |
| $2^{145430}-1, ..., 2^{145430}\cdot(2^{145430}-1)$ | $2^{72715}\cdot(2^{72715}-1)+2^{72715}-1$ | 145431 | 290880 |
| $2^{290860}-1, ..., 2^{290860}\cdot(2^{290860}-1)$ | $2^{145430}\cdot\ (2^{145430}-1)+\ 2^{145430}-1$ | 290861 | 581741 |
| $2^{581720}-1, ..., 2^{581720}\cdot(2^{581720}-1)$ | $2^{290860}\cdot(2^{290860}-1)+2^{290860}-1$ | 581721 | 1163462 |
| $2^{1163440}-1, ..., 2^{1163440}\cdot(2^{1163440}-1)$ | $2^{581720}\cdot(2^{581720}-1)+2^{581720}-1$ | 1163441 | 2326903 |
| $2^{2326880}-1, ..., 2^{2326880}\cdot(2^{2326880}-1)$ | $2^{1163440}\cdot(2^{1163440}-1)+2^{1163440}-1$ | 2326881 | 4653784 |
| $2^{4653760}-1, ..., 2^{25}\cdot(2^{4653760}-1)$ | $2^{2326880}\cdot(2^{2326880}-1)+2^{2326880}-1$ | 26 | 4653810 |
| $2^2(2^{4653783}-1), \cdots, 2^{4653760}(2^{4653783}-1)$ | $2^{25}\cdot(2^{4653760}-1)+2^2\cdot(2^{23}-1)$ | 4653759 | 9307569 |
| $2^{9307543}-1$ | $2_4^{4653760}\cdot(2^{4653783}-1)+2^{4653760}-1$ | 1 | 9307570 |

Computer calculations show that $l(9307543) = 29$ giving the following contradiction to exact equality in the Scholz-Brauer:

$$l(2^{9307543} - 1) \leq 9307570 < l(9307543) + 9307543 - 1 = 9307571$$

## References


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